

# Folding a 3D Euclidean Space

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## Abstract

This paper considers an extension of origami geometry to the case of “folding” a three dimensional (3D) space. The analysis is performed by defining a 3D extension of the so-called *axioms* of origami. In standard 2D folding, the axioms are elementary single-fold operations along a straight line which satisfy specific incidence constraints between given points and lines in the plane. The axioms were introduced over three decades ago by [Justin \(1986\)](#) and have been expressed under a variety of forms in studies of origami mathematics (e.g., [Alperin, 2000](#)). In the present extension, an infinite 3D Euclidean space is folded along a plane so as to satisfy specific incidence constraints between given points, lines and planes.

Following a previous study ([Lucero, 2017](#)), fold operations are analyzed by using the geometry of reflections. A spacial fold reflects all objects on each side of the fold plane onto the other side. Then, a total of twelve different 3D incidence constraints may be defined, and their formalization is shown in Table I. The table also shows the codimension of each constraint, which is the number of degrees of freedom that the constraint consumes when it is satisfied.

A 3D elementary fold operation is defined as a minimal set of constraints which have a total codimension not smaller than 3. Each of the incidences 1, 2, 4 and 12 have codimension 3 and hence each of them already define an elementary operation. Other operations may be defined by combining (a) two constraints of codimensions 1 and 2 respectively (16 combinations), (b) two constraints both of codimension 2 (10 combinations), or (c) three constraints of codimension 1 (20 combinations). The total of possible combinations is 50; however, some of them must be excluded: constraint 11 may not be used three times for the same operation or in combination with constraint 9, and constraint 9 may not be used twice. As an example, Fig. 1 illustrates the fold operation resulting from combining constraints 6 and 10: given a point  $P$ , a line  $m$  and a plane  $\pi$ , fold space along a plane that contains  $m$  so as to place  $P$  onto  $\pi$ .

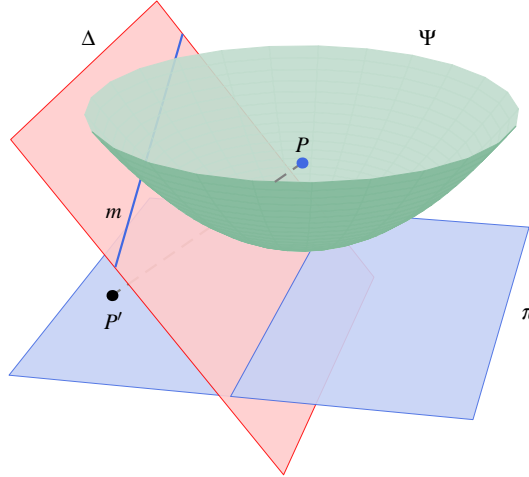
This paper will analyze and illustrate the geometry of the twelve 3D constraints and will explore solutions to relevant 3D fold operations. Potential applications of the results might include studies of Universe structure in cosmology ([Neyrinck, 2014](#)).

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**Table 1:** 3D incidence constraints.  $\mathcal{F}_\Delta(\cdot)$  denotes the image of an object by reflection on a plane  $\Delta$ ;  $P$  and  $Q$  are points;  $m$  and  $n$  are lines;  $\pi$  and  $\tau$  are planes.

Constraint	Definition	Codimension
1	$\mathcal{F}_\Delta(P) = Q$ , with $P \neq Q$	3
2	$\mathcal{F}_\Delta(m) = n$ , with $m \neq n$	3
3	$\mathcal{F}_\Delta(m) \cap n \neq \emptyset$ , with $m \cap n = \emptyset$	1
4	$\mathcal{F}_\Delta(\pi) = \tau$ , with $\pi \neq \tau$	3
5	$\mathcal{F}_\Delta(P) \in m$ , with $P \notin m$	2
6	$\mathcal{F}_\Delta(P) \in \pi$ , with $P \notin \pi$	1
7	$\mathcal{F}_\Delta(m) \subset \pi$ , with $m \not\subset \pi$	2
8	$\mathcal{F}_\Delta(P) = P$	1
9	$\mathcal{F}_\Delta(m) = m$ , and $\exists P \in m$ , $\mathcal{F}_\Delta(P) \neq P$	2
10	$\mathcal{F}_\Delta(m) = m$ , and $\forall P \in m$ , $\mathcal{F}_\Delta(P) = P$	2
11	$\mathcal{F}_\Delta(\pi) = \pi$ , and $\exists P \in \pi$ , $\mathcal{F}_\Delta(P) \neq P$	1
12	$\mathcal{F}_\Delta(\pi) = \pi$ , and $\forall P \in \pi$ , $\mathcal{F}_\Delta(P) = P$	3



**Figure 1:** Example of a 3D fold operation: the fold plane  $\Delta$  contains line  $m$  and reflects point  $P$  onto  $P'$  in plane  $\pi$ . Note that  $\Delta$  is tangent to paraboloid  $\Psi$ , with focus  $P$  and directrix plane  $\pi$ .

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